Classification of minimal Lorentzian surfaces in $\mathbb{S}^4_2(1)$ with constant Gaussian and normal curvatures

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Abstract

In this paper we consider Lorentzian surfaces in the 4-dimensional pseudo-Riemannian sphere $\mathbb{S}^4_2(1)$ with index 2 of curvature one. We obtain the complete classification of minimal Lorentzian surfaces $\mathbb{S}^4_2(1)$ whose Gaussian and normal curvatures are constants. We conclude that such surfaces have the Gaussian curvature 1/3 and the absolute value of normal curvature 2/3. We also give some explicit examples.

Keywords. Gaussian curvature, minimal submanifolds, Lorentzian surfaces, normal curvature

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1 Introduction

Surfaces with zero mean curvature play an important role on several branches of physics, mathematics as well as differential geometry. Classifications of minimal surfaces with constant Gaussian curvature in Riemannian spaces of constant curvature have been studied in a number of papers, [1, 12, 13, 15]. Also, a similar classification was considered for surfaces in pseudo-Riemannian spaces of constant curvature in [4, 7, 16, 17, 18].

One of the first important results in this direction was obtained by Pinl in [15], where he proved that there is no minimal surface with non-zero constant Gaussian curvature in a Euclidean space $\mathbb{E}^n$ of arbitrary dimension. Later, in [16] it was proved that this statement is still true if the ambient space is a Minkowski space $\mathbb{E}^n_1$ of arbitrary dimension.

On the other hand, if the ambient space is a (pseudo)-Riemannian space form with constant sectional curvatures $K_0 \neq 0$, then different results may occur in terms of existence of minimal surfaces with constant Gaussian curvature $K \neq K_0$. The Veronese surface and the Clifford torus in $\mathbb{S}^4(1)$ and the pseudo-Riemannian Clifford torus in the de Sitter space $\mathbb{S}^4_1(c)$, $c > 0$ are some of the most basic examples of minimal surfaces with constant Gaussian curvature. In [17], it was proved that a minimal surface with constant Gaussian curvature in $\mathbb{S}^4_1(c)$ is congruent to an open part of either a Clifford torus or a pseudo-Riemannian Clifford torus.

Further, in [18], Sakaki gave necessary and sufficient conditions for the existence of spacelike maximal surfaces in 4-dimensional pseudo-Riemannian space forms $\mathbb{S}^4_2(1)$ and $\mathbb{H}^4_2(-1)$ with

index 2, and he also obtained a characterization for maximal surfaces with constant Gaussian curvature in these space forms. In [7], Cheng gave a classification of complete maximal surfaces with constant scalar curvature in 4-dimensional pseudo-hyperbolic space $\mathbb{H}^4_2(c)$ with index 2 and of constant curvature $c < 0$.

In a recent paper, Chen obtained several classifications of minimal Lorentzian surfaces in arbitrary indefinite space forms, [5]. In particular, he obtained all minimal Lorentzian surfaces of constant curvature one in the pseudo Riemannian sphere $\mathbb{S}^n_t(1)$ of arbitrary dimension and index. In [5], he also proved that a minimal surface in a pseudo-Euclidean space $\mathbb{E}^n_t$ is congruent to a translation surface of two null curves. On the other hand, in [2] and [6], Chen and Yang gave the complete classification of flat quasi-minimal surfaces in the pseudo-Euclidean space $\mathbb{E}^4_2$.

Before we proceed, we want to point out to the minimal immersion from $\mathbb{S}^2(\frac{1}{3})$ into $\mathbb{S}^4(1)$ given by

$$\left(\frac{vw}{\sqrt{3}}, \frac{uw}{\sqrt{3}}, \frac{uv}{\sqrt{3}}, \frac{u^2-v^2}{\sqrt{3}}, \frac{u^2+v^2-2w^2}{6}\right), \quad u^2+v^2+w^2=3,$$

called the Veronese surface which has the following interesting property. It is well-known that a minimal parallel surface lying fully in $\mathbb{S}^4(1)$ is an open part of this surface, [8, 11]. The analogous of this result in the 4-dimensional pseudo-hyperbolic space $\mathbb{H}^4_2(-1)$ was obtained by Chen in [4]. He gave a minimal immersion of the hyperbolic plane $\mathbb{H}^2(-\frac{1}{3})$ of curvature $-1/3$ into $\mathbb{H}^4_2(-1)$ and he proved that, up to rigid motion of $\mathbb{H}^4_2(-1)$, this surface is the only parallel minimal surface lying fully in $\mathbb{H}^4_2(-1)$. Note that there is an immersion with zero mean curvature vector field from the de Sitter 2-space $\mathbb{S}^2_1(\frac{1}{3})$ of curvature $1/3$ into the pseudo-sphere $\mathbb{S}^4_2(1)$ with index 2 which is called the Lorentzian Veronese surface (see Example 2).

In this work, we study minimal Lorentzian surfaces in the 4-dimensional pseudo-sphere $\mathbb{S}^4_2(1)$. We obtain a characterization for minimal Lorentzian surfaces in $\mathbb{S}^4_2(1)$ with constant Gaussian curvature and constant normal curvature. We conclude that for such surfaces the Gaussian curvature is $1/3$ and the absolute value of the normal curvature is $2/3$. Also we obtain a characterization for minimal Lorentzian surfaces in $\mathbb{S}^4_2(1)$ that is congruent to the Lorentzian Veronese surface. Finally we give some explicit examples.

2 Prelimineries

Let M be a non-degenerated k-dimensional pseudo-Riemannian submanifold of an n-dimensional pseudo-Riemannian manifold N. We denote the Levi-Civita connections of N and M by $\widetilde{\nabla}$ and ∇, respectively. The Gauss and Weingarten formulas are given, respectively, by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$
$$\widetilde{\nabla}_X \xi = -A_\xi(X) + D_X\xi, \tag{2.2}$$

for any tangent vector field X, Y and any normal vector field ξ on M, where h and D are the second fundamental form and the normal connection of M in N, respectively, and A_ξ stands for the shape operator along the normal direction ξ. It is well-known that the shape operator A

and the second fundamental form h of M are related by

$$\langle A_\xi X, Y\rangle \;=\; \langle h(X,Y),\xi\rangle. \tag{2.3}$$

The mean curvature vector field of M in N is defined by

$$H = \frac{1}{k}\mathrm{tr}h. \tag{2.4}$$

A submanifold M in N is called minimal if H vanishes identically. In particular, if M is a surface in N, i.e., $k=2$, the Gaussian curvature K of M is defined by

$$K \;=\; \frac{R(X,Y,Y,X)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}, \tag{2.5}$$

where X, Y span the tangent bundle of M. A surface M is said to be flat if $K \equiv 0$ on M.

Let $\mathbb{E}^n_t$ denote the pseudo-Euclidean n-space with the canonical pseudo-Euclidean metric tensor of index t given by

$$g = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n} dx_j^2,$$

where $(x_1, x_2, \ldots, x_n)$ is a rectangular coordinate system of $\mathbb{E}^n_t$.

A non-zero vector v in $\mathbb{E}^n_t$ is called space-like, time-like or null (light-like) if $\langle v,v\rangle > 0$, $\langle v,v\rangle < 0$ or $\langle v,v\rangle = 0$, respectively.

We put

$$\begin{aligned} \mathbb{S}^{n-1}_t(r^2) &= \{v \in \mathbb{E}^n_t : \langle v,v\rangle = r^{-2}\},\\ \mathbb{H}^{n-1}_t(-r^2) &= \{v \in \mathbb{E}^n_{t-1} : \langle v,v\rangle = -r^{-2}\}, \end{aligned}$$

where $\langle\ ,\ \rangle$ is the indefinite inner product on $\mathbb{E}^n_t$, [14]. Here $\mathbb{S}^{n-1}_t(r^2)$ and $\mathbb{H}^{n-1}_t(-r^2)$ are complete pseudo-Riemannian manifolds of index t and of constant curvature r^2 and $-r^2$, respectively.

Furthermore, the light cone $\mathcal{LC}$ of $\mathbb{E}^n_t$ is defined by

$$\mathcal{LC} = \{v \in \mathbb{E}^n_t : \langle v,v\rangle = 0\}.$$

In the rest of the paper, we put $N = \mathbb{E}^n_t$. Then, Gauss, Codazzi and Ricci equations become

$$R(X,Y,Z,W) \;=\; \langle h(Y,Z), h(X,W)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.6a}$$
$$(\hat\nabla_X h)(Y,Z) \;=\; (\hat\nabla_Y h)(X,Z), \tag{2.6b}$$
$$\langle R^D(X,Y)\xi,\eta\rangle \;=\; \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.6c}$$

respectively, where R, R^D are the curvature tensors associated with the connections ∇ and D, respectively, and

$$(\hat\nabla_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

3 Minimal Lorentzian surfaces with constant Gaussian and normal curvatures

In this section we obtain complete classification of minimal Lorentzian surfaces in pseudo-sphere $\mathbb{S}^4_2(1)$ with constant Gaussian and normal curvatures.

First, we would like to state the following lemma obtained in [3] (see also [9, Proposition 2.1] and [10]).

Lemma 3.1. *[3] Locally there exists a coordinate system (u, v) on a Lorentzian surface M such that the metric tensor is given by*

$$g = -m^2(du \otimes dv + dv \otimes du),$$

for some positive smooth function $m = m(u, v)$. Moreover, the Levi-Civita connection of M is given by

$$\nabla_{\partial_u}\partial_u = \frac{2m_u}{m}\partial_u, \quad \nabla_{\partial_u}\partial_v = 0, \quad \nabla_{\partial_v}\partial_v = \frac{2m_v}{m}\partial_v \tag{3.1}$$

and the Gaussian curvature of M becomes

$$K = \frac{2(mm_{st} - m_s m_t)}{m^4}. \tag{3.2}$$

Let M be a Lorentzian surface in the pseudo-Riemannian space form $\mathbb{S}^4_2(1)$. We consider a local pseudo-orthonormal frame field $\{f_1, f_2; f_3, f_4\}$ of M such that $\langle f_1, f_2\rangle = \langle f_3, f_4\rangle = -1$ and $\langle f_A, f_B\rangle = 0$ for other cases. Then, by using (2.4) one can see that the mean curvature vector $\hat{H}$ in $\mathbb{S}^4_2(1)$ becomes

$$\hat{H} \quad = \quad -\hat{h}(f_1, f_2), \tag{3.3}$$

where $\hat{h}$ denote the second fundamental form of M in $S^4_2(1)$. On the other hand, the normal curvature $K^{\hat{D}}$ of M in $\mathbb{S}^4_2(1)$ is defined by

$$K^{\hat{D}} \quad = \quad -R^{\hat{D}}(f_1, f_2; f_3, f_4), \tag{3.4}$$

where $\hat{D}$ denote the normal curvature of M in $S^4_2(1)$. In the rest of the paper, by the abuse of notation, we put $K^D = K^{\hat{D}}$. Let x denote the position vector of M in $\mathbb{E}^5_2$. We will denote connection forms of M associated with the frame field under consideration by ω_A^B, $A, B = 1, 2, 3, 4$ which are defined by

$$\widetilde{\nabla}_X f_A = \sum_{B=1}^{4} \omega_A^B(X) f_B - \langle X, f_A\rangle\, x$$

for a vector field X tangent to M. By considering (2.3), one can check that connection forms satisfy

$$\begin{aligned} &\omega_1^3 = -\omega_4^2, \quad \omega_2^3 = -\omega_4^1, \quad \omega_1^4 = -\omega_3^2, \quad \omega_2^4 = -\omega_3^1 \\ &\omega_1^1 = -\omega_2^2, \quad \omega_3^3 = -\omega_4^4, \quad \omega_1^2 = \omega_2^1 = \omega_3^4 = \omega_4^3 = 0. \end{aligned} \tag{3.5}$$

Remark 1. By considering the local orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ given by $e_1 = (f_1 - f_2)/\sqrt{2}$, $e_2 = (f_1 + f_2)/\sqrt{2}$, $e_3 = (f_3 + f_4)/\sqrt{2}$ and $e_4 = (f_3 - f_4)/\sqrt{2}$, one can see that (3.4) becomes

$$K^D = R^{\hat{D}}(e_1, e_2; e_3, e_4). \tag{3.6}$$

3.1 Connection forms of minimal Lorentzian surfaces

In this subsection, we would like to focus on minimal Lorentzian surfaces and consider their connection forms.

Let M be a Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$ with the Gaussian curvature K and the normal curvature K^D, and let x be its position vector in $\mathbb{E}^5_2$. Then, by employing in Lemma 3.1, we see that tangent vector fields $f_1 = m^{-1}\partial_u$ and $f_2 = m^{-1}\partial_v$ form a local pseudo-orthonormal frame field for the tangent bundle of M. Because of (3.1), we have

$$\nabla_{f_i} f_1 = \phi_i f_1, \qquad \nabla_{f_i} f_2 = -\phi_i f_2, \tag{3.7}$$

where we put

$$\phi_1 = \omega_1^1(f_1) = \frac{m_u}{m^2} \qquad \phi_2 = \omega_1^1(f_2) = -\frac{m_v}{m^2}. \tag{3.8}$$

On the other hand, since M is a Lorentzian surface, its normal bundle in $\mathbb{S}^4_2(1)$ is spanned by two null vector fields f_3, f_4 such that $\langle f_3, f_4\rangle = -1$. Also, we put $f_5 = x$.

Now, we assume that M is minimal in $\mathbb{S}^4_2(1)$. Then, (3.3) implies $\hat{H} = -\hat{h}(f_1, f_2) = 0$, where $\hat{H}$ denote the mean curvature vector of M in $\mathbb{S}^4_2(1)$. On the other hand, since $\widetilde{\nabla}_{f_i} x = f_i$ we have $Df_5 = 0$ and $A_5 = -I$, where A_μ denotes the shape opearator along the normal vector field f_μ, $\mu = 3, 4, 5$. Thus, we have

$$D_{f_i} f_3 = \psi_i f_3, \qquad \nabla_{f_i} f_4 = -\psi_i f_4, \tag{3.9}$$

where we put $\psi_i = \omega_3^3(f_i),\ i = 1, 2$.

Therefore, by using (2.3), we obtain $\langle h(f_i, f_i), f_5\rangle = 0$ and $\langle h(f_1, f_2), f_5\rangle = 1$. Hence, we have

$$h(f_1, f_1) = -h^4_{11} f_3 - h^3_{11} f_4, \tag{3.10a}$$
$$h(f_1, f_2) = f_5, \tag{3.10b}$$
$$h(f_2, f_2) = -h^4_{22} f_3 - h^3_{22} f_4, \tag{3.10c}$$

where $h^\mu_{ij} = \langle h(f_i, f_j), f_\mu\rangle,\ i, j = 1, 2,\ \mu = 3, 4$. In this case, (2.3) implies

$$A_3(f_1) = -h^3_{11} f_2, \qquad A_3(f_2) = -h^3_{22} f_1, \tag{3.11a}$$
$$A_4(f_1) = -h^4_{11} f_2, \qquad A_4(f_2) = -h^4_{22} f_1. \tag{3.11b}$$

Moreover, by combining (2.5) and (3.10) with the Gauss equation (2.6a), we see that the Gaussian curvature of M takes the form

$$K = h^3_{22}h^4_{11} + h^3_{11}h^4_{22} + 1 \tag{3.12}$$

and the normal curvature of M becomes

$$K^D = h^3_{22}h^4_{11} - h^3_{11}h^4_{22} \tag{3.13}$$

because of the Ricci equation (2.6c), (3.4) and (3.11). We would like to state the following lemma that we will use later.

Lemma 3.2. *Let M be a minimal Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$. Assume that there exists a null tangent vector field X such that $h(X, X)$ is null. Then, K is constant if and only if K^D is constant.*

Proof. By replacing indices if necessary, we may assume that X is proportional to f_1 which implies either $h^4_{11} = 0$ or $h^3_{11} = 0$. These two cases imply either $K = h^3_{11}h^4_{22} + 1$, $K^D = -h^3_{11}h^4_{22}$ or $K = h^3_{22}h^4_{11} + 1$, $K^D = h^3_{22}h^4_{11}$, respectively. Hence, the proof follows. □

By a direct computation using the Codazzi equation (2.6b) and the Ricci equation (2.6c), one can obtain the following integrability conditions

$$\begin{aligned} f_2(h^4_{11}) &= (-\psi_2 + 2\phi_2)h^4_{11}, & (3.14a)\\ f_2(h^3_{11}) &= (\psi_2 + 2\phi_2)h^3_{11}, & (3.14b)\\ f_1(h^4_{22}) &= (-\psi_1 - 2\phi_1)h^4_{22}, & (3.14c)\\ f_1(h^3_{22}) &= (\psi_1 - 2\phi_1)h^3_{22}, & (3.14d)\\ K^D = h^3_{22}h^4_{11} - h^3_{11}h^4_{22} &= f_1(\psi_2) - f_2(\psi_1) + \phi_1\psi_2 + \phi_1\psi_1. & (3.14e) \end{aligned}$$

We will use the following lemma which directly follows from (3.14).

Lemma 3.3. *Let M be a flat minimal Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$. If the normal curvature K^D is constant, then it must be zero.*

Proof. Since $K = 0$, by re-defining u, v necessarily, we may assume $m = 1$ which implies $f_1 = \partial_u$, $f_2 = \partial_v$ and $\phi_1 = \phi_2 = 0$. Thus, (3.14e) becomes

$$K^D = (\psi_2)_u - (\psi_1)_v. \tag{3.15}$$

Now, we assume K^D is a non-zero constant. Note that if $h^4_{11}h^3_{22} = h^3_{11}h^4_{22} = 0$, then (3.13) implies $K^D = 0$ which is not possible. Therefore, without loss of generality, we may assume $h^4_{11}h^3_{22} \neq 0$. In this case, since K^D is constant, (3.12) and (3.13) imply that $h^4_{11}h^3_{22} = \text{const} \neq 0$. Therefore, from (3.14a) and (3.14d) we get

$$\psi_2 = -\Big(\ln|h^4_{11}|\Big)_v \quad \text{and} \quad \psi_1 = \Big(\ln|h^3_{22}|\Big)_u = -\Big(\ln|h^4_{11}|\Big)_u,$$

respectively. Hence, these two equations imply $(\psi_1)_v = (\psi_2)_u$. Thus, (3.15) gives $K^D = 0$ which yields a contradiction. □

3.2 The main result

In this subsection, we determine a necessary condition for a minimal Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$ having constant Gaussian and normal curvatures. First, we obtain a necessary condition.

Proposition 3.4. *Let M be a minimal Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$. If K and $K^D \neq 0$ are constants, then M is congruent to the surface given by*

$$x(s,t) = \frac{s^2}{2}\alpha(t) + s\beta(t) + \gamma(t) \tag{3.16}$$

for some smooth $\mathbb{E}^5_2$-valued maps α, β, and γ such that the induced metric takes the form

$$g = -(ds \otimes dt + dt \otimes ds) + 2\widetilde{m}dt \otimes dt, \tag{3.17}$$

for a smooth function $\widetilde{m}$.

Proof. If K and $K^D \neq 0$ are constant, then (3.12) and (3.13) imply $h^4_{11}h^3_{22} = \lambda$ and $h^3_{11}h^4_{22} = \nu$ for some constants λ, ν. Note that if $\lambda = 0$ and $\nu = 0$, then (3.13) implies $K^D = 0$ which is a contradiction. Therefore, without loss of generality, we may assume $\lambda \neq 0$. In this case, (3.14a) and (3.14d) imply

$$\begin{aligned} f_2(h^3_{22}) &= (\psi_2 - 2\phi_2)h^3_{22}, & (3.18)\\ f_1(h^4_{11}) &= (-\psi_1 + 2\phi_1)h^4_{11}, & (3.19) \end{aligned}$$

respectively. We will study the cases $\nu = 0$ and $\nu \neq 0$ separately.

Case I. $\nu \neq 0$. Then, (3.14b) and (3.14c) imply

$$\begin{aligned} f_2(h^4_{22}) &= (-\psi_2 - 2\phi_2)h^4_{22}, & (3.20)\\ f_1(h^3_{11}) &= (\psi_1 + 2\phi_1)h^3_{11}, & (3.21) \end{aligned}$$

respectively. By combining (3.19) with (3.21) and (3.14a) with (3.14b), we obtain

$$\phi_1 = \frac{1}{4}f_1(\ln|h^3_{11}h^4_{11}|) \quad \text{and} \quad \phi_2 = \frac{1}{4}f_2(\ln|h^3_{11}h^4_{11}|),$$

respectively. By combining these equations with (3.8), we get

$$-\partial_v(\ln m) = \partial_v(\ln|h^4_{11}h^3_{11}|) \quad \text{and} \quad \partial_u(\ln m) = \partial_u(\ln|h^4_{11}h^3_{11}|).$$

These two equations imply $(\ln m)_{uv} = 0$. Therefore, (3.2) yields $K = 0$, i.e., M is flat. Hence, Lemma 3.3 implies $K^D = 0$ which is a contradiction.

Case II. $\nu = 0$. By re-arranging f_1 and f_2 if necessary, we may assume $h^3_{11} = 0$. In this case, (3.10a), (3.11a) and (3.13) imply

$$\begin{aligned} h(f_1, f_1) &= -h^4_{11}f_3, & (3.22a)\\ A_3(f_1) = 0, \qquad & A_3(f_2) = h^3_{22}f_1, & (3.22b)\\ K^D &= h^4_{11}h^3_{22}. & (3.22c) \end{aligned}$$

Therefore, by combining Weingarten formula (2.2) with (3.22b), we obtain $\widetilde{\nabla}_{f_1} f_3 = \psi_1 f_3$ or, equivalently,

$$\frac{\partial}{\partial u} f_3 = m\psi_1 f_3. \tag{3.23}$$

By using (3.19), (3.22a) and (3.23), we get

$$\frac{\partial}{\partial u} h(f_1, f_1) = 2\frac{m_u}{m} h(f_1, f_1)$$

which implies

$$h(f_1, f_1) = h^4_{11} f_3 = m^2 \alpha(v). \tag{3.24}$$

for a $\mathbb{E}^5_2$-valued map α. Note that if $\alpha'(v) = 0$, then f_3 is parallel. However, since the codimension of M in $\mathbb{S}^4_2(1)$ is 2, the existence of a parallel normal vector field yields $K^D = 0$ which is a contradiction. Therefore, we have $\alpha' \neq 0$.

Now we define a local coordinate system (s, t) on M by

$$s = s(u, v) = \int_{u_0}^{u} m^2(\xi, v) d\xi, \quad t = v.$$

Then we have

$$\partial_u = m^2 \partial_s \quad \text{and} \quad \partial_v = \widetilde{m}\partial_s + \partial_t \tag{3.25}$$

which give

$$\langle \partial_s, \partial_s \rangle = 0, \quad \langle \partial_s, \partial_t \rangle = -1, \quad \langle \partial_t, \partial_t \rangle = 2\widetilde{m},$$

where $\widetilde{m} = \frac{\partial}{\partial v}\left(\int_{u_0}^{u} m^2(\xi, v) d\xi\right)$. Therefore, we obtain (3.17).

By a further computation using (3.17), we obtain $\nabla_{\partial_s}\partial_s = 0$. By combining this equation with (3.24) and (3.25) we get

$$\widetilde{\nabla}_{\partial_s}\partial_s = x_{ss} = \alpha(t).$$

By integrating this equation, we obtain (3.16) for some $\mathbb{E}^5_2$-valued maps β and γ. Hence, we completed the proof. □

Next, we obtain the complete classification of minimal Lorentzian surfaces in $\mathbb{S}^4_2(1)$ with constant Gaussian curvature and non-zero constant normal curvature.

Theorem 3.5. *Let M be a Lorentzian surface lying fully in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$. Then, M is minimal in $\mathbb{S}^4_2(1)$ with the constant Gaussian curvature K and non-zero constant normal curvature K^D if and only if it is congruent to the surface given by*

$$x(s, t) = \left(\frac{1}{2}s^2 + \frac{27}{40}\langle \alpha'''(t), \alpha'''(t)\rangle\right)\alpha(t) + \frac{3}{2}s\alpha'(t) + \frac{3}{2}\alpha''(t), \tag{3.26}$$

where α is a null curve in the light cone $\mathcal{LC}$ of $\mathbb{E}^5_2$ satisfying

$$\langle \alpha''(t), \alpha''(t)\rangle = \frac{4}{9}. \tag{3.27}$$

Proof. Assume that M is a minimal Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$ with the constant Gaussian curvature K and non-zero constant normal curvature K^D. Then, Proposition 3.4 implies that M is congruent to (3.16) for some smooth $\mathbb{E}^5_2$-valued maps $\alpha,\ \beta,\ \gamma$ such that the induced metric takes the form (3.17).

Then, by a simple computation using (3.17), we see that the Levi-Civita connection of M satisfies

$$\begin{aligned} \nabla_{\partial_s}\partial_s &= 0, & (3.28a)\\ \nabla_{\partial_s}\partial_t = \nabla_{\partial_t}\partial_s &= -\widetilde{m}_s\partial_s, & (3.28b)\\ \nabla_{\partial_t}\partial_t &= \widetilde{m}_s\partial_t + (2\widetilde{m}\widetilde{m}_s - \widetilde{m}_t)\partial_s. & (3.28c)\end{aligned}$$

Further, by using (2.5), (3.17) and (3.28), we obtain the Gaussian curvature of M as

$$K = \widetilde{m}_{ss}. \tag{3.29}$$

Since K is constant, (3.29) implies

$$\widetilde{m}(s,t) = \frac{K}{2}s^2 + c_1(t)s + c_2(t) \tag{3.30}$$

for some smooth functions $c_1(t)$ and $c_2(t)$ defined on some open interval in $\mathbb{R}$.

Note that because of (3.17), we have $\langle x_s, x_s\rangle = 0$ and $\langle x_t, x_t\rangle = 2\widetilde{m}$. Therefore, by a simple computation considering $\langle x, x\rangle = 1$ and using (3.30), (3.16), we obtain

$$\begin{aligned} \langle\alpha,\alpha\rangle = \langle\alpha',\alpha'\rangle = 0, && (3.31a)\\ \langle\gamma,\gamma\rangle = 1 \qquad \langle\gamma',\gamma'\rangle = 2c_2. && (3.31b)\end{aligned}$$

Therefore, (3.31a) yields that α is a null curve in the light cone $\mathcal{LC}$ of $\mathbb{E}^5_2$. Also, (3.31a) implies

$$\begin{aligned} \langle\alpha,\alpha'\rangle = \langle\alpha,\alpha''\rangle = \langle\alpha',\alpha''\rangle = \langle\alpha,\alpha'''\rangle = 0, && (3.31c)\\ \langle\alpha'',\alpha''\rangle = -\langle\alpha',\alpha'''\rangle = \langle\alpha,\alpha^{(4)}\rangle. && (3.31d)\end{aligned}$$

On the other hand, the tangent vector fields $\widetilde{f_1} = \frac{1}{m}f_1 = \partial_s$ and $\widetilde{f_2} = mf_1 = \widetilde{m}\partial_s + \partial_t$ form a pseudo orthonormal base field for the tangent bundle of M. Because of (3.28a), we have $\nabla_{\widetilde{f_1}}\widetilde{f_1} = 0$ which implies $\nabla_{\widetilde{f_1}}\widetilde{f_2} = 0$. Therefore, considering (3.16), the second fundamental form h of M in $\mathbb{E}^5_2$ satisfies

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{f_1}}\widetilde{f_2} =& h(\widetilde{f_1},\widetilde{f_2}) = h(f_1,f_2)\\ =&\widetilde{m}_s x_s + \widetilde{m}x_{ss} + x_{ts}\\ =&3K\frac{s^2}{2}\alpha + s(2c_1(t)\alpha + K\beta + \alpha') + (c_2(t)\alpha + c_1(t)\beta + \beta').\end{aligned} \tag{3.32}$$

Since M is minimal, we have (3.10b). By combining (3.10b) and (3.16) with (3.32), we obtain

$$\frac{3Ks^2}{2}\alpha + s(2c_1(t)\alpha + K\beta + \alpha') + c_2(t)\alpha + c_1(t)\beta + \beta' = \frac{s^2}{2}\alpha + s\beta + \gamma$$

which gives

$$\begin{aligned}\alpha &= 3K\alpha && \text{(3.33a)}\\ \beta &= 2c_1\alpha + K\beta + \alpha' && \text{(3.33b)}\\ \gamma &= c_2\alpha + c_1\beta + \beta' && \text{(3.33c)}\end{aligned}$$

Since α is non-zero, (3.33a) implies $K = \frac{1}{3}$. Therefore, (3.33b) becomes

$$\beta = 3c_1\alpha + \frac{3}{2}\alpha'. \tag{3.34}$$

By combining (3.34) and (3.33c), we get

$$\gamma = \left(c_2 + 3c_1^2 + 3c_1'\right)\alpha + \frac{9}{2}c_1\alpha' + \frac{3}{2}\alpha'' \tag{3.35}$$

which implies

$$\gamma' = \left(c_2' + 6c_1c_1' + 3c_1''\right)\alpha + \left(\frac{15}{2}c_1' + c_2 + 3c_1^2\right)\alpha' + \frac{9}{2}c_1\alpha'' + \frac{3}{2}\alpha''' \tag{3.36}$$

By considering (3.31), from (3.35), we obtain

$$1 = \langle\gamma,\gamma\rangle = \frac{9}{4}\langle\alpha'',\alpha''\rangle$$

which gives (3.27).

On the other hand, by a direct computation using (3.31) and (3.36), we obtain

$$2c_2 = -10c_1' - \frac{4}{3}c_2 + 5c_1^2 + \frac{9}{4}\langle\alpha''',\alpha'''\rangle$$

which gives

$$c_2 = -3c_1' + \frac{3}{2}c_1^2 + \frac{27}{40}\langle\alpha''',\alpha'''\rangle \tag{3.37}$$

By using (3.37) in (3.35), we get

$$\gamma = \left(\frac{9}{2}c_1^2 + \frac{27}{40}\langle\alpha''',\alpha'''\rangle\right)\alpha + \frac{9}{2}c_1\alpha' + \frac{3}{2}\alpha'' \tag{3.38}$$

By combining (3.16), (3.34) and (3.38) we get

$$x(s,t) = \left(\frac{1}{2}(s+3c_1)^2 + \frac{27}{40}\langle\alpha''',\alpha'''\rangle\right)\alpha + \left(\frac{3}{2}(s+3c_1)\right)\alpha' + \frac{3}{2}\alpha''. \tag{3.39}$$

From the parametrization that we obtain for M in (3.39), we see that, without loss of generality, we may choose $c_1 = 0$ by re-defining s properly. Hence, we have (3.26) which proves the necessary condition.

Conversely, assume that M is given by (3.26) for a curve α described in the theorem. Then, we have (3.31a) and (3.31c). By a simple computation, we see that the induced metric g of M satisfies (3.17) for the smooth function

$$\widetilde{m} = \frac{1}{6}s^2 + \frac{27}{40}\langle \alpha'''(t), \alpha'''(t)\rangle,$$

which yields that M has constant Gaussian curvature because of (3.29). Furthermore, by considering (3.31a) and (3.31c), from (3.26) we get $\langle x, x\rangle = 1$, i.e., M lies in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$.

On the other hand, $\widetilde{f}_1 = \partial_s$ and $\widetilde{f}_2 = \widetilde{m}\partial_s + \partial_t$ satisfies $\nabla_{\widetilde{f}_1}\widetilde{f}_1 = \nabla_{\widetilde{f}_1}\widetilde{f}_2 = 0$ as described while proving the necessary condition. Therefore, we have

$$h(\widetilde{f}_1, \widetilde{f}_2) = \widetilde{\nabla}_{\widetilde{f}_1}\widetilde{f}_2 = \widetilde{m}_s x_s + \widetilde{m} x_{ss} + x_{ts}.$$

By a simple computation, we see that the right-hand side of the above equation is x. Hence, M is minimal in $\mathbb{S}^4_2(1)$.

Finally, we have $\widetilde{\nabla}_{\widetilde{f}_1}\widetilde{f}_1 = h(\widetilde{f}_1, \widetilde{f}_1) = x_{ss} = \alpha(t)$. Therefore, for the null tangent vector field $X = \widetilde{f}_1$ we have $h(X, X)$ is null. Since K is constant and M is minimal in $\mathbb{S}^4_2(1)$, Lemma 3.2 implies that K^D is constant which completes the proof. ☐

3.3 Conclusions

In this subsection, we investigate some special cases and give some explicit examples.

Let M be the minimal surface given by (3.26) for a null curve α lying in the light cone $\mathcal{LC}$ of $\mathbb{E}^5_2$ satisfying (3.27). We consider the pseudo-orthonormal frame field $\{\widetilde{f}_1, \widetilde{f}_2; f_3, f_4\}$, where $\widetilde{f}_1$ and $\widetilde{f}_2$ are tangent vector fields described in the proof Theorem 3.5 and

$$\begin{aligned}
f_3 =& \alpha(t),\\
f_4 =& \frac{1}{2400}\Big(- 100s^4 - 162\left(5s^2\eta + 10s\eta' + 81\eta^2\right) + 6075\xi\Big)\alpha(t)\\
&+ \frac{1}{160}\Big(- 40s^3 - 270s\eta - 567\eta'\Big)\alpha'(t) - \frac{3}{20}\Big(5s^2 + 27\eta\Big)\alpha''(t)\\
&- \frac{3s}{4}\alpha'''(t) - \frac{9}{4}\alpha^{(4)}(t)
\end{aligned}$$

for the functions $\eta = \langle \alpha'''(t), \alpha'''(t)\rangle$ and $\xi = \langle \alpha^{(4)}(t), \alpha^{(4)}(t)\rangle$. By a direct computation, we obtain the Levi-Civita connection of M as

$$\nabla_{\widetilde{f}_1}\widetilde{f}_1 = \nabla_{\widetilde{f}_1}\widetilde{f}_2 = 0, \quad \nabla_{\widetilde{f}_2}\widetilde{f}_1 = -\frac{s}{3}\widetilde{f}_1, \quad \nabla_{\widetilde{f}_2}\widetilde{f}_2 = \frac{s}{3}\widetilde{f}_2 \tag{3.40}$$

and the second fundemental form of M as

$$h(\widetilde{f}_1, \widetilde{f}_1) = f_3, \quad h(\widetilde{f}_1, \widetilde{f}_2) = x, \quad h(\widetilde{f}_2, \widetilde{f}_2) = \left(\frac{27}{40}\eta''(t) - \frac{5103}{1600}\eta^2(t) + \frac{27}{16}\xi(t)\right) f_3 - \frac{2}{3}f_4. \tag{3.41}$$

In addition, the normal connection of M satisfies

$$D_{\widetilde{f}_1} f_3 = D_{\widetilde{f}_1} f_4 = 0, \quad D_{\widetilde{f}_2} f_3 = -\frac{2s}{3} f_3, \quad D_{\widetilde{f}_2} f_4 = \frac{2s}{3} f_4. \tag{3.42}$$

Therefore, we have

Corollary 3.6. *Let M be an oriented minimal Lorentzian surface in $\mathbb{S}^4_2(1) \subset \mathbb{E}^5_2$ with the Gaussian curvature K and normal curvature K^D. If K and $K^D \neq 0$ are constant, then $K = \frac{1}{3}$ and $|K^D| = \frac{2}{3}$.*

On the other hand, by combining (3.40)-(3.42), we obtain connection forms of M associated with the frame field $\{\widetilde{f}_1, \widetilde{f}_2, f_3, f_4\}$ as

$$\begin{aligned} \omega_3^3 = 2\omega_1^1 &= -\frac{2s}{3}, & \omega_1^4 &= 0, \quad \omega_1^3 = -\omega_1 \\ \omega_2^3 &= \left(\frac{27}{40}\eta''(t) - \frac{5103}{1600}\eta^2(t) + \frac{27}{16}\xi(t)\right)\omega_2, & \omega_2^4 &= -\frac{2}{3}\omega_2, \end{aligned} \tag{3.43}$$

where ω_1 and ω_2 are dual forms defined by $\omega_i(f_j) = \delta_{ij}$.

Example 2. [17] Let (x, y, z) be the natural coordinate system of $\mathbb{E}^3_1$ and $(u_1, u_2, u_3, u_4, u_5)$ that of $\mathbb{E}^5_2$. The mapping $\mathbf{x} : \mathbb{S}^2_1\left(\frac{1}{3}\right) \longrightarrow \mathbb{S}^4_2$ of the de Sitter space $\mathbb{S}^2_1\left(\frac{1}{3}\right)$ of curvature 1/3 into the pseudo-sphere $\mathbb{S}^4_2$ defined by

$$\begin{aligned} &u_1 = \frac{1}{6}(x^2 + y^2 + 2z^2), \;\; u_2 = \frac{1}{2\sqrt{3}}(x^2 - y^2), \;\; u_3 = \frac{1}{\sqrt{3}}xy, \\ &u_4 = \frac{1}{\sqrt{3}}xz, \;\; u_5 = \frac{1}{\sqrt{3}}yz \end{aligned}$$

is an isometric immersion of $\mathbb{S}^2_1\left(\frac{1}{3}\right)$ which is called the *Lorentzian Veronese surface*. A parametrization of the Lorentzian Veronese surface M_1 is given as

$$\begin{aligned} \mathbf{x}(u, v) = &\left(\frac{3}{2}\cosh^2\left(\frac{u}{\sqrt{3}}\right) - 1, \frac{\sqrt{3}}{2}\cosh^2\left(\frac{u}{\sqrt{3}}\right)\cos\left(\frac{2v}{\sqrt{3}}\right), \frac{\sqrt{3}}{2}\cosh^2\left(\frac{u}{\sqrt{3}}\right)\sin\left(\frac{2v}{\sqrt{3}}\right), \right. \\ &\left. \frac{\sqrt{3}}{2}\sinh\left(\frac{2u}{\sqrt{3}}\right)\cos\left(\frac{v}{\sqrt{3}}\right), \frac{\sqrt{3}}{2}\sinh\left(\frac{2u}{\sqrt{3}}\right)\sin\left(\frac{v}{\sqrt{3}}\right)\right). \end{aligned} \tag{3.44}$$

It can be proved that this surface is minimal in $\mathbb{S}^4_2(1)$. Moreover, it has constant normal curvature $K^D = -\frac{2}{3}$ and constant Gaussian curvature $K = \frac{1}{3}$.

Proposition 3.7. *Let M be the surface given by* (3.26) *for a null curve $\alpha(t)$ in the light cone $\mathcal{LC}$ of $\mathbb{E}^5_2$ satisfying* (3.27)*. If α satisfies*

$$\frac{27}{40}\eta''(t) - \frac{5103}{1600}\eta^2(t) + \frac{27}{16}\xi(t) = 0, \tag{3.45}$$

where $\eta = \langle \alpha'''(t), \alpha'''(t)\rangle$ and $\xi = \langle \alpha^{(4)}(t), \alpha^{(4)}(t)\rangle$, then M is congruent to the Lorentzian Veronese surface.

Proof. Let M_1 be Lorentzian Veronese surface given by (3.44) and M a surface described in Theorem 3.5 for a curve α. With the notation described in Sec. 3.1, we consider the orthonormal frame field $\{e_1, e_2; e_3, e_4\}$ given by

$$e_1 = \frac{\partial}{\partial u}, \quad e_2 = \operatorname{sech}\left(\frac{u}{\sqrt{3}}\right)\frac{\partial}{\partial v}, \quad e_3 = \sqrt{3}\hat{h}(e_1, e_1), \quad e_4 = \sqrt{3}\hat{h}(e_1, e_2) \tag{3.46}$$

satisfying $\varepsilon_1 = \langle e_1, e_1\rangle = -1$, $\varepsilon_2 = \langle e_2, e_2\rangle = 1$, $\varepsilon_3 = \langle e_3, e_3\rangle = 1$ and $\varepsilon_4 = \langle e_4, e_4\rangle = -1$. We put

$$\check{f}_1 = \zeta(e_1 - e_2), \quad \check{f}_2 = \frac{1}{2\zeta}(e_1 + e_2), \quad \check{f}_3 = \frac{2\sqrt{3}\zeta^2}{3}(e_3 - e_4), \quad \check{f}_4 = -\frac{\sqrt{3}}{4\zeta^2}(e_3 + e_4),$$

where ζ is a non-vanishing function satisfying

$$e_1(\zeta) - e_2(\zeta) = -\frac{\sqrt{3}\zeta}{3}\tanh\left(\frac{u}{\sqrt{3}}\right).$$

Then the null vector fields $\check{f}_1, \check{f}_2, \check{f}_3, \check{f}_4$ form a pseudo-orthonormal frame field for M. Furthermore, by a direct computation, we see that connection forms corresponding to this frame field satisfy

$$\begin{aligned} \check{\omega}_3^3 = 2\check{\omega}_1^1 = -\frac{2\check{s}}{3}, &\qquad \check{\omega}_1^4 = 0, \quad \check{\omega}_1^3 = -\check{\omega}_1 \\ \check{\omega}_2^3 = 0, &\qquad \check{\omega}_2^4 = -\frac{2}{3}\check{\omega}_2 \end{aligned} \tag{3.47}$$

for the coordinate function $\check{s}$ given by

$$\check{s} = \frac{\sqrt{3}}{2\zeta}\tanh\left(\frac{u}{\sqrt{3}}\right) - \frac{3}{2\zeta^2}(e_1(\zeta) + e_2(\zeta)).$$

By comparing (3.43) and (3.47), we see that if α satisfies (3.45), then the connection forms of M_1 corresponding to the frame field $\{\check{f}_1, \check{f}_2, \check{f}_3, \check{f}_4\}$ coincides with that of M corresponding to frame field $\{\widetilde{f}_1, \widetilde{f}_2, f_3, f_4\}$. Hence, we obtain that M is congruent to M_1 if (3.45) is satisfied. $\square$

In the next example, by considering Proposition 3.7, we obtain a parametrization of a Lorentzian surface which is congruent to the Lorentzian Veronese surface.

Example 3. We consider the null curve

$$\alpha(t) = \frac{1}{3\sqrt{3}}\left(2\cos t, 2\sin t, \cos 2t, \sin 2t, \sqrt{3}\right)$$

in the light cone $\mathcal{LC}$ of $\mathbb{E}^5_2$. The, for this α (3.26) gives an explicit example of minimal surface in $\mathbb{S}^4_2(1)$ with constant Gaussian and normal curvatures. Since α satisfies (3.45), the Lorentzian surface given by

$$\begin{aligned} x(s,t) =& \frac{1}{6\sqrt{3}}\Big(2s(s\cos t - 3\sin t), 2s(s\sin t + 3\cos t), \left(s^2 - 9\right)\cos 2t - 6s\sin 2t, \\ & \left(s^2 - 9\right)\sin 2t + 6s\cos 2t, \sqrt{3}\left(s^2 + 3\right)\Big). \end{aligned} \tag{3.48}$$

is congruent to the Lorentzian Veronese surface.

Remark 4. By considering the definition of the coordinate function s in the proof of Proposition 3.4, we would like to conclude that the new paramatrization of the Lorentzian Veronese surface presented in (3.48) possesses the following interesting property: The parameter curve $x(s_0,t)$ is a null geodesic of the Lorentzian Veronese surface for any constant s_0.

In the following example, we obtain a minimal surface which is not congruent to Lorentzian Veronese surface.

Example 5. In this example, we consider the curve

$$\alpha_0(t) = \frac{1}{3\sqrt{3}}\left(\cos 2t \cot t, 2\cos^2 t, \cos t \cot t \cos\left(\sqrt{3}\ln(\tan t + \sec t)\right),\right.$$
$$\left.\cos t \cot t \sin\left(\sqrt{3}\ln(\tan t + \sec t)\right), \cos t\right)$$

for $0 < t < \frac{\pi}{2}$ and the surface M given by (3.26) for $\alpha = \alpha_0$. By a direct computation, we obtain

$$\begin{aligned} h(\widetilde{f_1}, \widetilde{f_1}) =& f_3, \\ h(\widetilde{f_1}, \widetilde{f_2}) =& x, \\ h(\widetilde{f_2}, \widetilde{f_2}) =& \left(\frac{21}{800}\left(-180\cos 2t + 45\cos 4t - 121\right)\csc^4 t \sec^4 t\right) f_3 - \frac{2}{3} f_4, \end{aligned}$$

where $\widetilde{f_1}, \widetilde{f_2}$ are the tangent vector fields described above and $f_3 = \alpha_0(t)$. Thus, M is a minimal surface in $\mathbb{S}^4_2(1)$ with constant Gaussian and normal curvatures.

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